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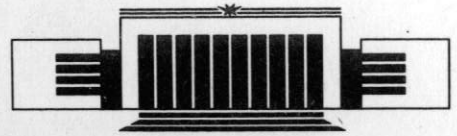

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Quantum Chaos and Peculiarities of Diffusion in Wigner Representation

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ABSTRACT

Peculiarities of energy diffusion in the well-known model of a quantum rotator excited by an external periodic δ-like field are investigated in Wigner representation. It is shown that in semi-classical region of parameters the quantum diffusion can be approximately described by means of classical mappings in discrete phase space. Both the phenomenon of quantum limitation of classical diffusion and the quantum resonance for which the mean energy is increasing quadratically in time manifests itself the motion of such classical models.

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1. INTRODUCTION

Much attention has been recently paid to the study of quantum systems which are chaotic in the classical limit (see, for instance, [1—12]). It is due to the fact that chaos is a typical phenomenon in classical dynamical nonlinear systems, even with a few degree of freedom and the development of quantum-mechanical methods for the analysis of such system is an important and urgent task. Investigation of the quantum chaos in simple models [1, 3, 12] showed that even in deep semi-classical region quantum effects can play an essential role. Main phenomenon is a restriction of chaotic diffusion which takes place in the corresponding classical systems. This phenomenon can be important, for example, in the case of strong excitation of quantum systems in semi-classical region by external coherent fields (Hydrogen atom [13, 14], moleculas [15], etc.).

The present paper concerns the investigation of peculiarities of energy diffusion in intensively studied simple model of quantum stochastisity—a quantum rotator excited by a periodic sequence of δ-impulses:

$$\hat{H}=\frac{\gamma\hbar^2\hat{n}^2}{2}+\varepsilon f(\theta)\sum_{t=-\infty}^{\infty}\delta(\tau-Tt)\,;\qquad \hat{n}=-i\frac{\partial}{\partial\theta}. \tag{1.1}$$

The model (1.1) contains itself the most interesting peculiarities of dynamical behaviour of quantum systems which are chaotic in the classical limit ($\hbar=0$). Such a model can be approximately

describe a real physical system with nonequidistant unperturbed spectrum under strong external field which has a large number of harmonics. Here θ is the phase variable, $\hat{I}=\hbar\hat{n}$ is the operator of momentum, γ^{-1} is momentum coefficient, ε and T — the strength and period of perturbation. In what follows we assume $f(\theta+2\pi)=f(\theta)$ with $|f(\theta)|\sim 1$.

For the first time the system (1.1) with $f(\theta)=\cos\theta$ was numerically investigated in [3]. The main result [3] consists in the discovery of considerable differences in behaviour of a quantum system in comparison with the classical one in the case when classical motion is chaotic with a characteristic linear increase of mean energy of the system. In particular, the rate of the energy growths corresponds to the classical one only during some time $t<t^*$, which depends essentially on the parameters of the system; then it sharply drops. Further investigation of peculiarities of such behaviour was made in [4] where it was shown analytically that for rational value of parameter ζ:

$$\zeta=\frac{\gamma\hbar T}{2\pi} \tag{1.2}$$

asymptotic form E_t is quadratic: $E_t\sim t^2$ for $t\to\infty$. In such notation (1.2) the value $\zeta=2$ corresponds to the so-called main quantum resonance in Refs [3, 4].

The results of numerical investigations of the main energy increase for the quantum rotator can be formulated in the following way. When the value of parameter ζ is equal to a «good» number $\zeta=r/q$ (r and q are not too large) the mean energy increase has strongly quadratic dependence starting from $t=0$. When the rational number ζ is worsened, $r, q\to\infty$; $r/q=\text{const}$ (it corresponds to the transition to the irrational value ζ) the law of mean energy increase shows characteristic limitation of classical diffusion for relatively large times and only after this time asymptotic quadratic behaviour starts to manifest itself (see, also [11]). It is interesting to note, that in the numerical experiments we always have the case of quantum resonance because of the finite digits of all numbers. However, the behaviour of the system is to be corresponding to the nonresonant case during enormous large time. It can be shown [5] that the wave functions for two close values of parameter ζ differ by a small value $\sim\delta\zeta=|\zeta_1-\zeta_2|$:

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\,|\psi_{\zeta_1}(\theta,t)-\psi_{\zeta_2}(\theta,t)|^2<\varkappa^2 t^3\delta\zeta;\qquad \varkappa=\varepsilon/\hbar. \tag{1.3}$$

Therefore, for our purpose one can confine oneself to the analysis of the system (1.1) only for rational values ζ. The case of rational values of ζ (1.2) can be also of special interest since it corresponds to resonances

$$\omega_n\equiv\frac{E_{n+1}-E_n}{\hbar}=\frac{(2n+1)\,r\Omega}{q};\qquad (E_n=\gamma\hbar^2n^2/2;\quad \Omega=2\pi/T)$$

between harmonics of frequency of unperturbed system and harmonics of that external force.

There are a number of works in which the above mentioned peculiarities of dynamics of a quantum rotator is investigated from different points of view (see, for example, [5—12]). The present paper is devoted to the investigation of the system (1.1) on the basis of formalism of Wigner function. The advantage of Wigner's formulation of quantum mechanics consists in the fact that evolution of the system occurs, by analogy with the classical case, in the phase plane (see, for example, rewiev [16]). Therefore, for the purpose of comparison of classical and quantum dynamics the Wigner representation suits mostly and allows, to our opinion, to explain characteristic peculiarities of the behaviour of a quantum rotator using semi-classical language.

The structure of this paper is the following. Main formulae and statements concerning Wigner representation for the quantum rotator which partially contains Ref. [11] are given in the next section. It is shown that Wigner representation naturally leads to the «classical model of quantum stochastisity» which has been phenomenologically introduced in [10] for the account of influence of the discreteness of phase space on system's dynamics. Section 3 contains analytical and numerical investigation of the properties of such classical model of quantum stochastisity. In the last Section 4 the results concerning the law of the mean energy increase for the system (1.1) and for the classical models are compared.

2. WIGNER REPRESENTATION FOR THE QUANTUM ROTATOR

Let $\hat{A}$ be an arbitrary operator which is a function of an angle variable and of operator $\hat{n}=-i\partial/\partial\theta$; $\hat{A}\equiv A(\theta,\hat{n})=A(\theta+2\pi,\hat{n})$. Then one can determine mutually single-valued transformation of the operator $\hat{A}$ to c-number function that will be called a transform of the operator $\hat{A}$:

$$A(\theta,\hat{n})\leftrightarrow a(\varphi,p)\,;\quad 0\leqslant\varphi<2\pi\,;\quad p=0,\pm1,\ldots \tag{2.1}$$

The convolution of a product of two operators has the form:

$$\mathrm{Tr}\,[A(\theta,\hat{n})\,B(\theta,\hat{n})]=\sum_{p=-\infty}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,a(\varphi,p)\cdot b(\varphi,p)\,. \tag{2.2}$$

Evolution of the quantum rotator will be described by a density matrix $\hat{\rho}$ at time moment immediately before the t-th kick. Using (2.1) and an explicit form of the Hamiltonian (1.1) one can obtain an equation for evolution of Wigner function (a transform of matrix density) for one period of motion. The mapping has the simplest analytical form in the case of a choice of $f(\theta)=\cos 2\theta$ which will be analysed in what follows:

$$\rho_{t+1}(\varphi,p)=\sum_{p'=-\infty}^{\infty}\int_0^{2\pi}d\varphi'\,\tilde{\delta}(\varphi'+2\pi\zeta p-\varphi)\,J_{p-p'}(2\varkappa\,\sin 2\varphi')\,\rho_t(\varphi',p')\,. \tag{2.3}$$

Here ζ and $\varkappa$ are determined in (1.2), (1.3); $J_p(z)$ are Bessel functions; $\tilde{\delta}$ is a periodic δ-function with a period 2π. The state of the rotator with a definite value p will be chosen as the initial condition which corresponds to

$$\rho_0(\varphi,p)=\frac{1}{2\pi}\delta_{p,p_0}\,.$$

Note two circumstances which follows immediately from (2.3). First of all, the mapping of Wigner function is local in phase and nonlocal in action. Secondly, in the quantum case due to discreteness of phase space in action $I=\hbar p$ one can choose invariant countable sets

$$\varphi_n=\{\varphi_0+2\pi\zeta\,n\}_{2\pi}\,;\quad n=0,\pm1,\ldots \tag{2.4}$$

where $\{\ldots\}_{2\pi}$ means a fractional part on modulo 2π, and φ_0 is an arbitrary initial phase (see Fig. 1). Evolution of Wigner function $\rho_t(\varphi,p)$ on a set $\{\varphi_n\}$ does not depend on values $\rho_t(\varphi,p)$ in different parts of phase space of measure 1. In the case when ζ is rational, $\zeta=r/q$, the sequence of phases $\{\varphi_n\}$ in (2.4) is finite ($0\leqslant n<q$). In the case when ζ is irrational the sequence $\{\varphi_n\}$ is infinite and densely covers an interval $0\leqslant\varphi<2\pi$. This difference leads to the difference in expressions for quantum-mechanical averages for an arbitrary operator $\langle A(t)\rangle=\mathrm{Tr}\,[\hat{A}\hat{\rho}_t]$. In particular, for $\zeta=r/q$ from (2.2) we have

$$\langle A(t)\rangle=\int_0^{2\pi/q}d\varphi_0\Big[\sum_{p=-\infty}^{\infty}\sum_{n=0}^{q-1}a(\varphi_n,p)\,\rho_t^{(\varphi_0)}(\varphi_n,p)\Big]\,. \tag{2.5}$$

Here $a(\varphi,p)$ is a transform of the operator $\hat{A}$, the upper index (φ_0) indicates an explicit dependence of Wigner function on the phase φ_0.

For our further purposes it is convenient to pass from relations (2.3) to dynamic equations of motion for p and φ. Introduce the following transformations:

$$\begin{aligned}p_{t+1}&=p_t+[\,2\varkappa\,\sin 2\varphi_t]_{int}+\Delta p_t\,;\\ \varphi_{t+1}&=\{\varphi_t+2\pi\,\zeta\,p_{t+1}\}_{2\pi}\,,\end{aligned} \tag{2.6}$$

where p_t dimensionless integer quantum action; operation $[\ldots]_{int}$ means that the integer part is taken; Δp_t is a discrete integer value distributed according to some specific law $W(\Delta p_t)$. Requirement of the discreteness of action variable dictates the presence of the operation $[\ldots]_{int}$ in (2.6) and introduction of an additional quasi-random term which reflects the fact of nonlocality of the mapping (2.3) in action. It is easy to see that for the mapping (2.6) to lead the evolution law (2.3) it is necessary to choose the distribution function $W(\Delta p_t)$ in the form

$$\begin{aligned}W(\Delta p_t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\xi\,\exp\{&-i\xi\Delta p_t-i\xi[\,2\varkappa\sin 2\varphi_t]_{int}+\\&+2i\varkappa\,\sin 2\varphi_t\cdot\sin\xi\}\,.\end{aligned} \tag{2.7}$$

For simplicity we confine ourselves to the case of $\varkappa\gg1$. Then

$$W(\Delta p_t) \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} d\xi \, \exp[-i\xi\Delta p_t - 2i\varkappa(\xi - \sin\xi)\cdot\sin 2\varphi_t] \equiv \tilde{W}(\Delta p_t) .$$

Function $\tilde{W}(\Delta p_t)$ has the following properties:

$$\sum_{\Delta p} \tilde{W}(\Delta p) = 1 ; \quad \sum_{\Delta p} \Delta p \, \tilde{W}(\Delta p) = 0 ; \quad \sum_{\Delta p} (\Delta p)^2 \, \tilde{W}(\Delta p) = 0 . \tag{2.8}$$

These expressions allow to assume that for $\varkappa \gg 1$ the contribution of the term Δp_t (see (2.6)) in the action diffusion can be considered small, and behaviour of the rotator is approximately described by the following mapping:

$$\begin{aligned} p_{t+1} &= p_t + [\, 2\varkappa \, \sin 2\varphi_t]_{int} , \\ \varphi_{t+1} &= \{\varphi_t + 2\pi\zeta p_{t+1}\}_{2\pi} . \end{aligned} \tag{2.9}$$

This model was phenomenologically introduced [10] for the account of influence of discreteness of phase space on the behaviour of the system. Numerical analysis [10] shows that the model (2.9) is capable of describing limitation of quantum diffusion qualitatively. Further, following [10] we shall refer to (2.5) as the «classical model of quantum stochastisity».

To estimate the influence of nonlocality in (2.6) we write the kernel of transformation in (2.3) as the sum of two terms $K_Q = K_{cq} + K_V$. Here

$$K_{cq}(\varphi, p \,|\, \varphi', p') = \tilde{\delta}(\varphi' + 2\pi\zeta p - \varphi) \, \delta_{p,\, p' + [\, 2\varkappa \sin 2\varphi']_{int}} \tag{2.10}$$

provided Wigner function evolution according to the classical model of quantum stochastisity (2.9), and $K_V = K_Q - K_{cq}$ is responsible for nonlocality of transformation of Wigner function (2.3) Then time evolution of the Wigner function can be written as

$$\rho_t = (\hat{K}_{cq} + \hat{K}_V)^t \, \rho_0 ; \quad \rho_0 = \frac{1}{2\pi} \delta_{p,0} \tag{2.11}$$

where $\hat{K}_{cq,\,V}\rho$ has the meaning

$$\hat{K}_{cq,\,V}\rho = \int_0^{2\pi} d\varphi' \sum_{p'} K_{cq,\,V}(p, \varphi \,|\, p', \varphi') \, \rho(p', \varphi') .$$

The product in (2.11) has 2^t terms which are the products of operators $\hat{K}_{cq}$ and $\hat{K}_V$ in different combinations. Therefore, the expression for mean energy of the system it is convenient to represent in the following form:

$$\begin{aligned} E_t = \frac{1}{2\pi} \int d\varphi \sum_p \frac{p^2}{2} [\, \hat{K}_{cq}^t + \hat{K}_{cq}^{t-1} \hat{K}_V + \\ + \hat{K}_{cq}^{t-2} (\hat{K}_V^2 + \hat{K}_V \hat{K}_{cq}) + \ldots + \hat{K}_V(\ldots)] \, \rho_0 . \end{aligned} \tag{2.12}$$

Using (2.8) it can be shown that at $\varkappa \gg 1$ the terms which have $\hat{K}_V$ on the right side are relatively small. It means that all of the terms in (2.12) are ordered by the value of contribution into mean energy. It should be mentioned that only the first term in (2.12) gives positive contribution. The rest terms may have arbitrary signs. To check these statements we have made the numerical calculations of mean energy of the system on the basis of the formula (2.12) with parameters $2\varkappa = 10.2$, $\zeta = 1/16$ for the first four steps ($t = 1, \ldots, 4$). The numerical data show that relative contribution into the mean energy of terms corresponding to different combinations of operators $\hat{K}_{cq}$ and $\hat{K}_V$ is in good agreement with the order of their position in the expression (2.12). In particular, the ratio of contributions of the last and the first terms in (2.12) has the order of 1%.

3. CLASSICAL MODEL OF QUANTUM STOCHASTICITY

The results of the previous section show that at $\varkappa \gg 1$ the energy increase for the quantum rotator is mainly determined by the properties of the mapping for $\hat{K}_{cq}$. Pass on to the analysis of the classical model of quantum stochastisity (2.9). In what follows we shall investigate the behaviour of the system (1.1) at rational values $\zeta = r/q$. The case of irrational ζ will be considered as transition $r, q \to \infty$. As it was mentioned in Section 2, for rational values ζ in (2.9) the trajectory of each particle is on the invariant set (see Fig. 1) consisting of q points of phase φ. Since all the values of a phase φ are numbered it is convenient to rewrite the mapping (2.9) in the form:

$$\begin{aligned} p_{t+1} &= p_t + \left[2\varkappa \, \sin 2\left(\varphi_0 + 2\pi \frac{r}{q} n_t \right) \right]_{int} , \\ n_{t+1} &= [\, n_t + p_{t+1} \,]_q . \end{aligned} \tag{3.1}$$

In (3.1) n_t is discrete phase ($0 \leqslant n_t < q$) on the t-th step of transformation, $[...]_q$ is the integer part modulo q. It is easy to see that the mappings (3.1) are invariant with respect to the shift $n \to n+q$ and $p \to p+q$. Thus, motion of the particle can be considered on a torus consisting of $N = q \times q$ points, and any trajectory can be characterized by two parameters. The first is period M — the number of steps before the locking of trajectory (since the number of points on a torus is finite any trajectory for $M \leqslant N$ steps will lock). The second parameter is the number of rotations J round torus along the axis of action which the trajectory has made before locking. As examples two trajectories locked on a torus with the number of rotations 1 and 0 are represented in Fig. 1.

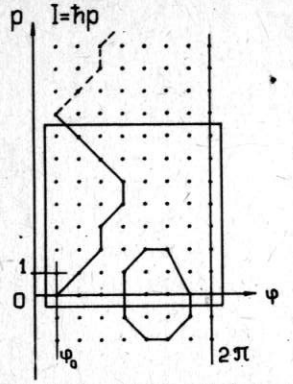

Fig. 1. Invariant set (represented by points) for the parameter value $\zeta = 1/8$. As an example, two trajectories of (2.9) locked on torus with the number of rotations equal to 1 and 0 are shown.

Let us estimate characteristic values of parameters M and J. Two limitting cases are possible: a) $2\varkappa \gtrsim q$ and b) $2\varkappa \ll q$. In the first case any point of a torus is accessible for a particle at every step. Supposing that the particle moves randomly in the discrete phase space it is easy to estimate probability that the trajectory will not lock for M steps

$$P(M) \approx \exp(-M^2/2N); \quad M/N \ll 1. \tag{3.2}$$

Assuming $P(M) \approx 0.5$, we have a characteristic period of trajectories

$$M^* = \sqrt{N} = q. \tag{3.3}$$

It can be shown that in this case trajectories with nonzero number of rotations J necessarily exist, the relative number of these trajectories being sufficiently large.

Suppose now we make a transition to the irrational value ζ increasing values r and q. Then the second case is to be realized: $2\varkappa \ll q$. It means that not the whole region of a torus is accessible for a particle at every step of transformations. Assuming that the classical diffusion occurs along the axis of action we can estimate the number of points of phase space occupied by the motion of the

system: $\tilde{N} \approx \varkappa q \sqrt{2t}$. Then for the time of locking $M_1^* \sim \sqrt{\tilde{N}}$ we have

$$M_1^* \sim (\varkappa q)^{2/3}. \tag{3.4}$$

The estimate (3.4) is essentially some upper estimate since it is based on the assumption of accessibility of the considered region of phase space at each step of transformation. In this case the probability for the trajectory to be locked without any rotation round torus is increased (in comparison with the first case). The condition for realization of such phenomenon can be represented in the form $\varkappa \sqrt{2M_1^*} < q$. Substituting value M_1^* from (3.4) we find the critical relation

$$2\varkappa^2 < q. \tag{3.5}$$

If this relation takes place then the majority of trajectories is locked on a torus without any rotation ($J = 0$).

Existence of the critical relation (3.5) for parameters $\varkappa$ and q is illustrated in Fig. 2 ($a-c$) where trajectories of $q = 101$ particles on a torus with the initial distribution $p_0 = 0$ are represented. Value of the parameter ζ is equal $\zeta = 10/101$. In case of Fig. 2,*a* ($2\varkappa = 5.0$) there is no trajectory with the rotations on a torus along the axis of action. Fig. 2,*b* corresponds to value $2\varkappa = 10.0$, in this case six trajectories have made one rotation. In the case of Fig. 2,*c* $2\varkappa = 20.0$ four trajectories have value $J = \pm 1$, fourteen — $J = \pm 2$, two — $J = \pm 3$.

Pass on to the analysis of the mean energy increase for the system (2.9). Consider the set of particles uniformly distributed on the interval $[0, 2\pi]$. Choose an arbitrary point from the set. Due to the fact that the term $[2\varkappa \sin \varphi_t]_{int}$ represents itself a step function, and locking of the trajectory occurs on a torus after a finite number of steps we can surround the chosen point by some small (but finite) region $\Delta\varphi_k$, any point from which move according to the same law. Thus, the time dependence for mean energy of the system $E_t = \langle p_t^2 \rangle / 2$ can be represented in the form

$$E_t = \frac{1}{4\pi} \sum_k \Delta\varphi_k \left[J_k q \frac{t}{M_k} + f_k(t) \right]^2; \quad \sum_k \Delta\varphi_k = 2\pi \tag{3.7}$$

where J is the number of rotations, M_k is a period of trajectory, $f_k(t)$ is some periodic function with a period M_k. Note, that in the numerical experiment such blocks $\Delta\varphi_k$ is well observed.

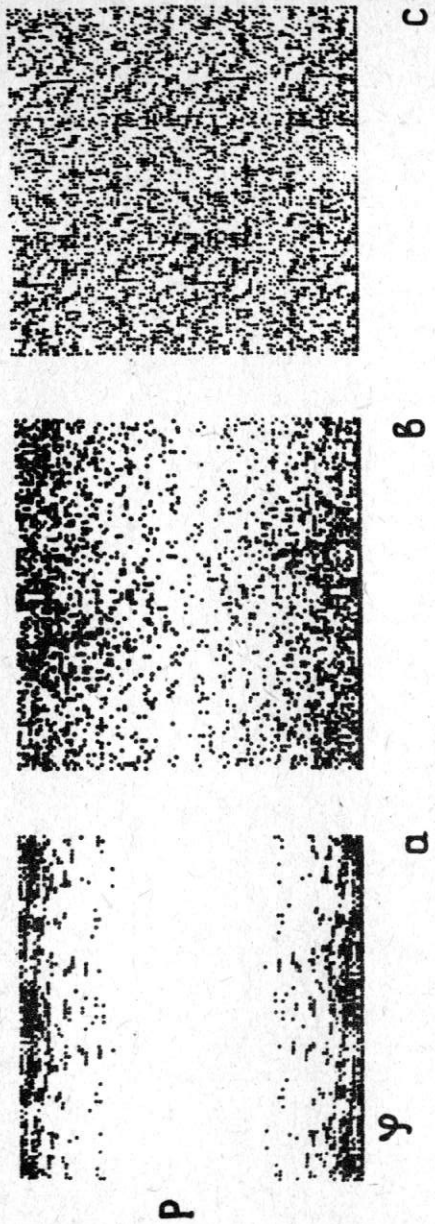

Fig. 2. Numerical data for mapping (3.1). Figures show trajectories on torus of $q=101$ particles «starting» from the bottom line. Values of parameters $\zeta=10/101$, $\varphi_0=0.0$; *a*) $2\varkappa=5.0$, *b*) $2\varkappa=10.0$, *c*) $2\varkappa=20.0$. In «*a*» case all trajectories have value $J=0$ (J is the number of rotations round torus along the axis of action); in «*b*» case six particles have $J=\pm1$; in «*c*» case four have $J=\pm1$, fourteen have $J=\pm2$, two have $J=\pm3$.

From (3.7) it follows that if the number of trajectories with zero values J_k is small the behaviour of E_t at finite times will be similar to quasi-periodic one. In the opposite case the law of energy increase will have strongly quadratic character. The number of points with nonzero values of J_k will apparently depend on relation between parameters of the system (3.5). Plots of increasing mean energy of the system are represented in Fig. 3 up to times $t=200$ for the values of parameters $2\varkappa=10.0$ and $2\varkappa=20.0$ and different values of parameter ζ (curves B). Values $\zeta=r/q$ are chosen to be proportional to the terms of the series of «golden mean»; thus, ζ tends to $(\sqrt{5}-1)/20$ with r, q increase. Averaging was made over the segment $[0, 2\pi]$ which consists of the system of two thousand uniformly distributed particles. To control the evolution of the system we increased the ensemble of averaging up to four thousand particles. This, practically, doesn't change the dependence E_t for times intervals under consideration. Numbers in figure correspond to the maximal value of energy on the vertical axis. It is seen from the figure that there occurs a change of a quadratic regime to that of «diffusion limitation».

Now we discuss the problem of the time scale t_d of diffusion limitation. As it was mentioned, time t_d has the meaning of limitation time of classical diffusion for the quantum system (1.1) only when ζ is the irrational number. However, in numerical experiments one can use the resonance case $\zeta=r/q$ at sufficiently large r and q. As it is seen from Fig. 3, plots E_t for $\zeta=377/6100$ and $\zeta=1597/25340$ do not practically differ on the considered time scale, and this indicates convergence of $E_t(\zeta_r)$ to $E_t(\zeta_{ir})$ at $\zeta_r=r/q\to\zeta_{ir}$ (existence of such a convergence for the quantum rotator (1.1) is guaranteed by the estimate (1.3)).

It is evident that for our model with discrete phase space time t_d is associated with the characteristic times of the locking of trajectories. The information about a period of locking trajectories can be obtained from the function $P(M)$ which represents itself probability of unlocking trajectory within time interval M. In the numerical experiment the function $P(M)$ was calculated as the ratio of trajectories which are not locked up to time moment M to their total number. The typical dependence $P(M)$ up to $M=400$ is represented in Fig. 4. Numerical experiment shows that the form of $P(M)$ does not change with r/q ($r/q\to\zeta_{ir}$) increasing; i. e. at a given value of ζ, $P(M)$ is a function of only $\varkappa$.

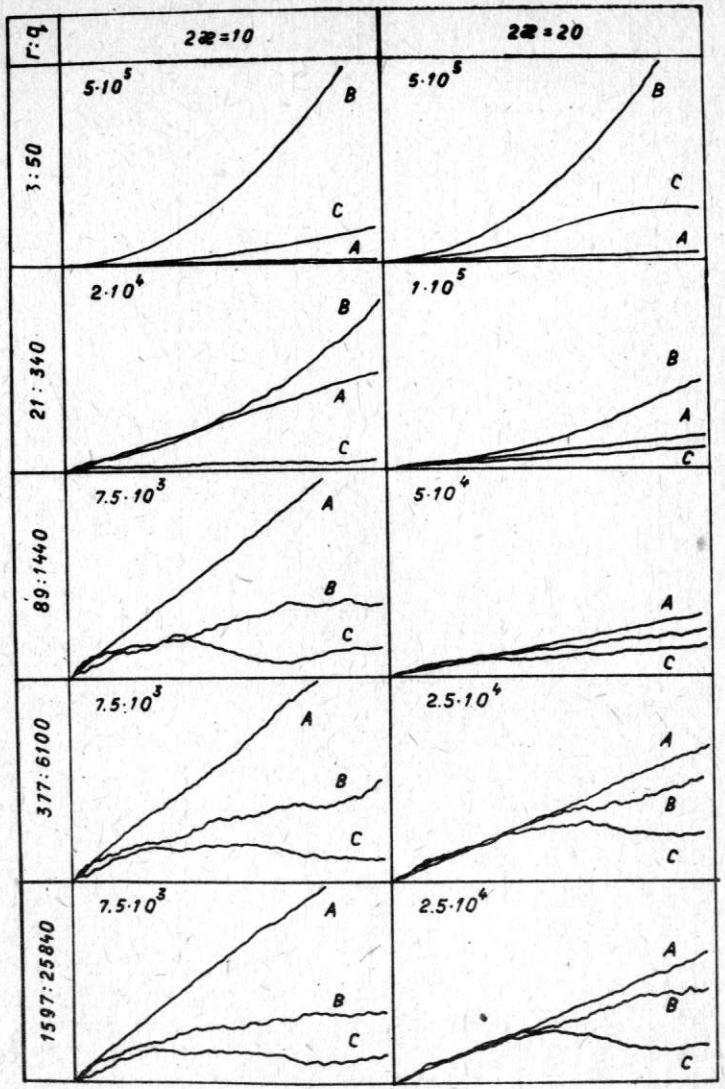

Fig. 3. Mean energy increase up to time $t=200$ for classical rotator (curves A), classical model of quantum stochastisity (curves B), quantum rotator (curves C). Numbers in figure show the scale of E_t along vertical axis.

When the dependence $P(M)$ is known it is easy to obtain the upper estimate for the law of increasing E_t. Supposing that the energy increase of a particle for time $t<M$ (M is a period of locking) is determined by the diffusion law and taking into account the fact that after times $t>M$ the particle under consideration does not contribute to E_t increase we have

$$E_t < \varkappa^2 \int_0^t P(M)\, dM . \tag{3.8}$$

Thus, E_t deviates from the classical linear diffusion law, and by analogy with a quantum case it is possible to introduce diffusion limitation time t_d for the model (2.9). As it is seen from Fig. 4, probability function $P(M)$ besides characteristic time M_1^* (3.4) (which, practically, corresponds to the length of a «tail» of the function $P(M)$) has another characteristic time M_2^* determining a sharp initial decrease of function $P(M)$. Time M_2^* does not depend on q and determines, according to (3.8), the diffusion limitation time t_d in the classical model of quantum stochastisity.

Rough estimate for time M_2^* can be obtained if we assume that the number of accessible points in phase φ of the order $\varkappa\sqrt{2M_2^*}$ is equal to the number of these points in p. In this case for the effective size of phase space we have $N^2 \sim (\varkappa\sqrt{2M_2^*})^2$ that under condition $M_2^* \sim \sqrt{N_2}$ gives

$$M_2^* \sim 2\varkappa^2 . \tag{3.9}$$

The estimate (3.9) agrees with that of diffusion limitation time t_d for the quantum rotator [10]. In conclusion we note that inequality (3.5) can also be obtained from the condition $M_1^* < M_2^*$.

4. CONCLUSION REMARKS

We represent some numerical results for the mean energy increase for a classical model of quantum stochastisity (3.1) comparing with the model of a quantum rotator (1.1) whose mean energy has been calculated by the formula

$$E_t = \frac{1}{4\pi} \int_0^{2\pi} \psi^*(\theta, t) \left(-\frac{\partial^2}{\partial\theta^2} \right) \psi(\theta, t)\, d\theta ; \qquad \psi(\theta, 0) = e^{i p_0 \theta} .$$

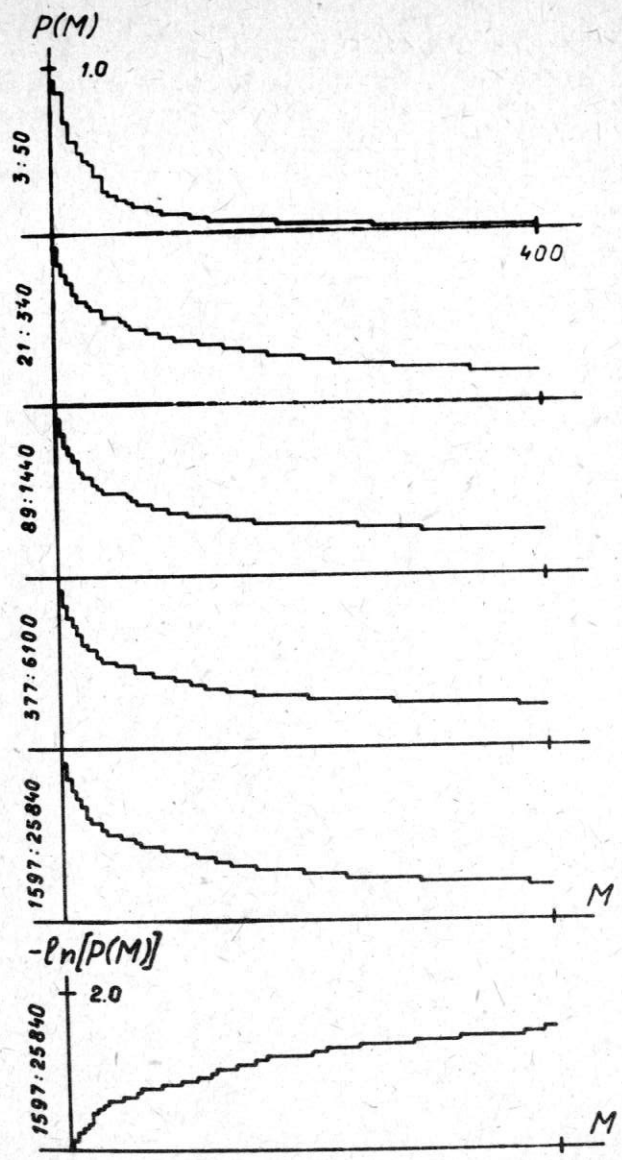

Fig. 4. Function $P(M)$ for the probability of unlocking trajectory for time M. Value of parameter $2\varkappa = 10.0$.

From Fig. 3 (curves B and C) it is clear that a classical model of quantum stochastisity describes approximately all peculiarities of energy diffusion in the initial quantum system (1.1). When ζ tends to the irrational number $(r, q \to \infty)$, and the condition $2\varkappa^2 \ll q$ is fulfilled, the system (3.1) reveals characteristic energy diffusion limitation by analogy with a quantum model (1.1), for both models diffusion limitation time coinciding by the order of magnitude.

It is important that the model (3.1) also describes a pure quantum phenomenon called in [3, 10] as quantum resonance at which mean energy has characteristic asymptotic form t^2. In Fig. 3 such type of motion is observed at $q = 50$ and 340 for time $t \leqslant 200$. Nevertheless, it is seen from Fig. 3 that curves B are situated in time above curves C. Such behaviour is explained by the structure of the series in (2.13) for the mean energy of a quantum rotator. As numerical simulation show the terms of the series containing operator $\hat{K}_V$ and responsible for nonlocality of a quantum mapping give in the sum negative contribution into mean energy. Numerical data for energy diffusion in the system of a classical rotator («standard mapping») are presented in Fig. 3 (curves A); they correspond to mapping (2.9) but with continuous action p (without operation $[...]_{int}$).

We have also carried out numerical experiments at the same values of parameters as in Fig. 3 but for the initial occupation of the first level of the system ($p_0 = 1$). Energy values E_t in comparison with the initial occupation of a zero level for time under consideration differ not more than two times, qualitative behaviour of the curves remaining the same. Apparently, for further comparison of diffusion in models (1.1) and (3.4) it makes sense to choose the initial distribution in the form of a packet of levels; this will allow to exclude large fluctuations which are connected with the initial conditions.

In conclusion note that the developed above approach for the study of dynamics of a quantum rotator (1.1) is, in fact, a new variant of semi-classical approximation for the case when in the classical limit ($\hbar = 0$) the motion is chaotic ($\varkappa = \varepsilon/\hbar \gg 1$; $K = 8\pi\varkappa\zeta \gg 1$; $\zeta \ll 1$). In such approach the trajectories are not usual classical trajectories of particle motion, but ones determined by the equations (2.9) (or (3.1)). Thus, we take into account the property of discreteness of phase space in action, and this enables to describe qualitatively such important property of the system as diffusion limitation. Further constructing of perturbation theory in the semi-classical re-

gion of parameters is connected with an account of terms in (2.13) containing operators $\hat{K}_V$.

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